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On Best Partial Bases

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1. INTRODUCTION

The *partial basis problem* is to determine a subsequence of a given length of a given "basis" for which best approximation is closest. This paper is a continuation of [6] and is based on written notes communicated privately by Allan Pinkus in response to the original version of [6]. We are deeply indebted to him.

Let X be a normed linear space, let f, $h_1, h_2,..., h_N \in X$ and let n be an integer, $1 \leqslant n < N$. For every subsequence $s = \{g_j\}_{j=1}^{n}$ of length n of $\{h_j\}_{j=1}^{N}$ consider

$$e(s) = \min \left\| f - \sum_{j=1}^{n} c_j g_j \right\|$$

where the minimum is taken over all possible choices of the scalars $c_1,..., c_n$. A *best partial basis of* $\{h_j\}_{j=1}^{N}$ *of length* n (to approximate f) is an s^* minimizing $e(s)$ among all the above subsequences s.

In Section 2 we obtain a condition guaranteeing that the initial segment $\{h_1, h_2,..., h_n\}$ of the "basis" $\{h_1, h_2,..., h_N\}$ of functions is the unique best partial basis of length n (to approximate f), and a condition guaranteeing that the terminal segment $\{h_{N-n+1}, h_{N-n+2},..., h_N\}$ is the (unique) best partial basis of length n. We also study the special basis $\{1, x,..., x^{N-1}\}$ of monomials. In Section 3 we give some examples. Section 4 is a generalization to extended complete Tchebycheff systems.

2. MAIN RESULTS

Tchebycheff systems play a key role in our results on the partial basis problem. A *Tchebycheff system*, or a *T-system* on a real interval I is a sequence $\{f_1, f_2,..., f_n\}$ of real functions defined on I such that whenever

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$x_1 < x_2 < \cdots < x_n$ and all $x_j \in I$, the determinant of the $n \times n$ matrix whose jth row ($j = 1, 2, \ldots, n$) is $f_j(x_1)\ f_j(x_2)\ \cdots\ f_j(x_n)$ is positive. An example of a Tchebycheff system on $(0, \infty)$ is $\{x^{\lambda_k}\}_{k=1}^n$ whenever $\lambda_1 < \lambda_2 < \cdots < \lambda_n$, n arbitrary [5, p. 9].

The basis for our subsequent theorems is the following result, part (a) of which is Theorem 1 of [6].

THEOREM 1. *Let*

$$\{h_1, h_2, \ldots, h_N, f\} \tag{1}$$

be a sequence of real functions, continuous on $[a, b]$ $(-\infty < a < b < \infty)$, *let* $1 \leqslant p \leqslant \infty$ *and let n be an integer,* $1 \leqslant n < N$. *Let* ε_n, ε_{n+1} *be each* 1 *or* -1.

(a) *Suppose, for* $k = n$, $n+1$, *every subsequence of* (1) *of length k, after multiplying its last element by* ε_k, *becomes a T-system on* $[a, b]$. *Then*

$$\{h_{N-n+1}, h_{N-n+2}, \ldots, h_N\}$$

is the unique best partial basis of $\{h_j\}_{j=1}^N$ *of length n to approximate f in the* $L^p(a, b)$ *norm.*

(b) *Suppose, for* $k = n$, $n+1$, *every subsequence of*

$$\{f, h_1, h_2, \ldots, h_N\} \tag{2}$$

of length k, after multiplying its last element by ε_k, *becomes a T-system on* $[a, b]$. *Then*

$$\{h_1, h_2, \ldots, h_n\}$$

is the unique best partial basis of $\{h_j\}_{j=1}^N$ *of length n to approximate f in the* $L^p(a, b)$ *norm.*

Proof of (b). Let k be n or $n+1$, let $\{g_j\}_{j=1}^k$ be a subsequence of $\{h_N, h_{N-1}, \ldots, h_1, f\}$ of length k and let

$$\varepsilon_k' = \varepsilon_k(-1)^{k(k-1)/2}.$$

If $a \leqslant x_1 < x_2 \cdots < x_k \leqslant b$, then since $\{g_{k+1-j}\}_{j=1}^k$ is a subsequence of $\{f, h_1, \ldots, h_N\}$ of length k,

$$\varepsilon_k' \begin{vmatrix} g_1(x_1) & \cdots & g_1(x_k) \\ \vdots & & \vdots \\ g_k(x_1) & \cdots & g_k(x_k) \end{vmatrix} = \varepsilon_k'(-1)^{k(k-1)/2} \begin{vmatrix} g_k(x_1) & \cdots & g_k(x_k) \\ \vdots & & \vdots \\ g_1(x_1) & \cdots & g_1(x_k) \end{vmatrix}$$

$$= \varepsilon_k \begin{vmatrix} g_k(x_1) & \cdots & g_k(x_k) \\ \vdots & & \vdots \\ g_1(x_1) & \cdots & g_1(x_k) \end{vmatrix} > 0.$$

By (a), $\{h_n, h_{n-1},..., h_1\}$ is the unique best partial basis of $\{h_{N+1-j}\}_{j=1}^{N}$ of length n to approximate f in the $L^p(a, b)$ norm. Hence $\{h_j\}_{j=1}^{n}$ is the unique best partial basis of $\{h_j\}_{j=1}^{N}$ of length n to approximate f in the $L^p(a, b)$ norm.

THEOREM 2. *Let n, N be integers, $1 \leqslant n < N$, and let $h_1, h_2,..., h_N, f$ be real functions defined on a real interval I.*

(A) *Suppose every "gapless" subsequence of*

$$\{h_1, h_2,..., h_N, f\} \tag{3}$$

(i.e., one consisting of consecutive terms) of length $\leqslant n+1$ is a T-system on I. Then every subsequence of (3) *of length $\leqslant n+1$ is a T-system on I. Hence, by Theorem* 1(a), *if $I = [a, b]$, $-\infty < a < b < \infty$, if $1 \leqslant p \leqslant \infty$ and if $h_1, h_2,..., h_N$, f are continuous on I, then $\{h_{N-n+1}, h_{N-n+2},..., h_N\}$ is the unique best partial basis of $\{h_j\}_{j=1}^{N}$ of length n to approximate f in the $L^p(a, b)$ norm.*

(B) *Suppose every "gapless" subsequence of*

$$\{f, h_1, h_2,..., h_N\} \tag{4}$$

of length $\leqslant n+1$ is a T-system on I. Then every subsequence of (4) *of length $\leqslant n+1$ is a T-system on I. Hence, by Theorem* 1(b), *under the hypotheses of the last sentence of* (A), *$\{h_1, h_2,..., h_n\}$ is the unique best partial basis of $\{h_j\}_{j=1}^{N}$ of length n to approximate f in the $L^p(a, b)$ norm.*

To prove Theorem 2 we need the following theorem for whose history, starting with M. Fekete [3, Lemma, p. 92], see [4, pp. 58, 59, 96].

THEOREM 3. *Let Φ be an $m \times k$ real matrix, $m \geqslant k \geqslant 1$. If, for $r = 1, 2,..., k$, the determinant of every $r \times r$ submatrix of Φ composed from the first r columns and some r consecutive rows of Φ is positive, then the determinant of every $k \times k$ submatrix of Φ is positive.*

Proof of Theorem 2(A). Let $\{g_j\}_{j=1}^{k}$ be a subsequence of $\{h_1, h_2,..., h_N, f\}$, $1 \leqslant k \leqslant n+1$, and let $x_1 < x_2 < \cdots < x_k$ be points of I. Set

$$\Phi = \begin{pmatrix} h_1(x_1) & \cdots & h_1(x_k) \\ \vdots & & \vdots \\ h_N(x_1) & \cdots & h_N(x_k) \\ f(x_1) & \cdots & f(x_k) \end{pmatrix}.$$

Then the hypothesis of the second sentence of Theorem 3 holds, because for $r=1, 2,..., k$, every gapless subsequence of $\{h_1,..., h_N, f\}$ of length r is a T-system on I. By Theorem 3,

$$\begin{vmatrix} g_1(x_1) & \cdots & g_1(x_k) \\ \vdots & & \vdots \\ g_k(x_1) & \cdots & g_k(x_k) \end{vmatrix} > 0.$$

Theorem 2(B) is similarly proved.

We shall need the following

LEMMA 4. *Let $-\infty < a < b < \infty$ and let g be a real function, continuous on $[a, b]$, with $g^{(k)}(x) > 0$ throughout (a, b) for some $k \geqslant 1$. Then $\{1, x,..., x^{k-1}, g(x)\}$ is a T-system on $[a, b]$.*

Proof. Let $a \leqslant x_1 < x_2 \cdots < x_{k+1} \leqslant b$. The inequality

$$\begin{vmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_{k+1} \\ \vdots & \vdots & & \vdots \\ x_1^{k-1} & x_2^{k-1} & \cdots & x_{k+1}^{k-1} \\ g(x_1) & g(x_2) & \cdots & g(x_{k+1}) \end{vmatrix} \neq 0$$

is an elementary, well known fact (see, e.g., [1], p. 77, Problem 8). To see that the last determinant, D, is positive, replace in it g by the function $tg(x) + (1-t)\, x^k$ and denote the resulting determinant $D(t)$, so that

$$D(t) \equiv tD + (1-t) \begin{vmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_{k+1} \\ \vdots & \vdots & & \vdots \\ x_1^k & x_2^k & \cdots & x_{k+1}^k \end{vmatrix} \equiv tD + (1-t) \prod_{1 \leqslant r < s \leqslant k+1} (x_s - x_r).$$

If $D < 0$, then for some $t \in (0, 1)$, $D(t) = 0$ which is impossible as $(tg(x) + (1-t)\, x^k)^{(k)} = tg^{(k)}(x) + k!(1-t) > 0$ throughout (a, b).

As an application of Theorem 2 and Lemma 4 we have

THEOREM 5. *Let $0 < a < b < \infty$; n, N integers, $1 \leqslant n < N$. Let f be a real function, positive and continuous on $[a, b]$ and assume that, for $k = 1, 2,..., n$, $(-1)^k f^{(k)}(x) > 0$ throughout (a, b). Let $1 \leqslant p \leqslant \infty$. Then $\{1, x,..., x^{n-1}\}$ is the unique best partial basis of $\{x^{j-1}\}_{j=1}^N$ of length n to approximate f in the $L^p(a, b)$ norm.*

Proof. By Lemma 4, for $k = 1, 2,..., n$, $\{1, x,..., x^{k-1}, (-1)^k f(x)\}$ is a

T-system on $[a, b]$ and, hence, so is $\{f(x), 1, x, \ldots, x^{k-1}\}$. The desired conclusion follows from the last sentence of Theorem 2(B).

The method of proof of Theorem 5 gives also an alternative proof of the following slightly weakened form of Theorem 4 of [6].

THEOREM 6. *Assume the first three sentences of Theorem 5, with* $(-1)^k f^{(k)}(x) > 0$ *replaced by* $[x^{k-N} f(x)]^{(k)} > 0$. *Then* $\{x^{N-n}, x^{N-n+1}, \ldots, x^{N-1}\}$ *is the unique best partial basis of* $\{x^{j-1}\}_{j=1}^{N}$ *of length n to approximate f in the* $L^p(a, b)$ *norm.*

Proof. By Lemma 4, for $k = 1, 2, \ldots, n$, $\{1, x, \ldots, x^{k-1}, x^{k-N}f(x)\}$ is a T-system on $[a, b]$, hence so is $\{x^{N-k}, x^{N-k+1}, \ldots, x^{N-1}, f(x)\}$. Apply now the last sentence of Theorem 2(A).

Remark 7. As noted in Remarks 6 of [6], our Theorem 1(a) continues to hold if $[a, b]$ is replaced by (a, b), if $1 \leqslant p \leqslant \infty$ is replaced by $1 \leqslant p < \infty$, and if $h_1, \ldots, h_N$, f belong to $L^p(a, b)$. It follows from its proof, that Theorem 1(b) continues, too, to hold under the corresponding modification. Consequently, the conclusions of Theorem 2(A) and 2(B) continue to hold if we replace the last sentence of Theorem 2(A) up to and including "on I," by "Hence, if $I = (a, b)$, $-\infty < a < b < \infty$, if $1 \leqslant p < \infty$ and if $h_1, \ldots, h_N$, f are continuous on I and belong to $L^p(I)$,". Clearly, Lemma 4 continues to hold if $[a, b]$ there is replaced by (a, b). Hence, Theorem 5 continues to hold if $0 < a < b < \infty$ is replaced by $0 \leqslant a < b < \infty$, and its second and third sentences by "Let f be a real function and suppose $(-1)^k f^{(k)}(x) > 0$ for $k = 0, 1, \ldots, n$ and every $x \in (a, b)$ and that $f \in L^p(a, b)$ for some $p \in [1, \infty)$."

3. EXAMPLES

Let $-\infty < a < b < \infty$, and let f be a real function. It is said to be *absolutely monotone* in (a, b) iff $f^{(k)}(x) \geqslant 0$ for $k = 0, 1, 2, \ldots$ and every $x \in (a, b)$. It is said to be *completely monotone* in (a, b) iff $(-1)^k f^{(k)}(x) \geqslant 0$ for $k = 0, 1, 2, \ldots$ and every $x \in (a, b)$, namely, iff $f(-x)$ is absolutely monotone in $(-b, -a)$.

COROLLARY 8. *Let n, N be integers,* $1 \leqslant n < N$, *and let f be a real function. Suppose either* (i) $0 \leqslant a < b < \infty$, $1 \leqslant p < \infty$, *and* $f \in L^p(a, b)$ *or* (ii) $0 < a < b < \infty$, $p = \infty$, *f is positive and continuous in* $[a, b]$. *Assume, in addition, that f is completely monotone in* (a, b). *Then*: I. *If f does not coincide in* (a, b) *with a polynomial of the form* $\sum_{\nu=0}^{n-1} a_\nu x^\nu$, *then* $\{1, x, x^2, \ldots, x^{n-1}\}$ *is the unique best partial basis of* $\{x^{j-1}\}_{j=1}^{N}$ *of length n to* *approximate f in the $L^p(a,b)$ norm.* II. *If f coincides in (a,b) with such a polynomial, then a subsequence $\{x^{\mu_j}\}_{j=1}^n$ of $\{x^{j-1}\}_{j=1}^N$ is a best partial basis of the latter of length n to approximate f in the $L^p(a,b)$ norm iff every integer ν, $0 \leqslant \nu \leqslant n-1$, with $a_\nu \neq 0$ equals some μ_j. Thus, if f coincides in (a,b) with such a polynomial with all $a_\nu \neq 0$, then $\{1, x, ..., x^{n-1}\}$ is the unique best partial basis of $\{x^{j-1}\}_{j=1}^N$ of length n to approximate f in the $L^p(a,b)$ norm.*

Proof. By hypothesis, $f(-x)$ is absolutely monotone in $(-b, -a)$. If, for some k, $0 \leqslant k \leqslant n$, $f^{(k)}$ vanishes somewhere in (a,b), then so does $g(x) \equiv [f(-x)]^{(k)}$ somewhere in $(-b, -a)$; since $g(x)$ is itself absolutely monotone in $(-b, -a)$, it vanishes identically there [7, Corollary 3a, p. 147], hence, so does $f^{(k)}(x)$ in (a,b), and therefore $f(x)$ coincides in (a,b) with a polynomial of the form $\sum_{\nu=0}^{n-1} a_\nu x^\nu$. Conclusion I of Corollary 8 follows from Theorem 5 and from the last sentence of Remark 7. Conclusion II is immediate.

Examples. Let n, N be integers, $1 \leqslant n < N$.

1. Suppose $c > 0$ and either (iii) $0 \leqslant a < b < \infty$, $1 \leqslant p < \infty$, or (iv) $0 < a < b < \infty$, $p = \infty$. Then by Corollary 8, $\{1, x, x^2, ..., x^{n-1}\}$ is the unique best partial basis of $\{x^{j-1}\}_{j=1}^N$ of length n to approximate e^{-cx} in the $L^p(a,b)$ norm.

2. Similarly, suppose either (iv) $0 \leqslant a < b \leqslant 1$, $1 \leqslant p < \infty$, or (v) $0 < a < b < 1$, $p = \infty$. Then by Corollary 8, $\{1, x, x^2, ..., x^{n-1}\}$ is the unique best partial basis of $\{x^{j-1}\}_{j=1}^N$ of length n to approximate $-\log x$ in the $L^p(a,b)$ norm.

3. Suppose $d > 0$ and either (vi) $0 < a < b < \infty$, $1 \leqslant p \leqslant \infty$, or (vii) $0 \leqslant a < b < \infty$, $1 \leqslant p < 1/d$. Then the same conclusion as in Examples 1 and 2 holds for the function x^{-d}.

4. Suppose $c > 0$, $0 < d < 1$ and either (viii) $0 \leqslant a < b \leqslant c^{1/d}$, $1 \leqslant p < \infty$, or (ix) $0 < a < b < c^{1/d}$, $p = \infty$. Then the same conclusion holds for the function $c - x^d$.

5. Suppose $a_0, a_1, ..., a_r, b_0, b_1, ..., b_s$ are reals, $r < s$, $a_r b_s > 0$. One readily sees that there is an $\hat{a} \geqslant 0$ such that, throughout $(\hat{a}, \infty)$, $\sum_{j=0}^s b_j x^j \neq 0$ and, in fact, setting there

$$\rho(x) = \sum_{j=0}^r a_j x^j \Big/ \sum_{j=0}^s b_j x^j,$$

we have

$$(-1)^k \rho^{(k)}(x) > 0 \qquad \text{for} \quad k = 0, 1, ..., n \text{ and every } x > \hat{a}.$$

Let $\hat{a} < a < b < \infty$, $1 \leqslant p \leqslant \infty$. By Theorem 5, the conclusion of Examples 1–4 holds for $\rho(x)$.

Finally, here is an example for Theorem 6.

6. Let $c > 0$ and observe that, for $k = 1, 2, \ldots, n$, we have on $(0, \infty)$,

$$(x^{k-N}e^{cx})^{(k)} = \sum_{j=0}^{k} \binom{k}{j} j! \binom{k-N}{j} x^{k-N-j} c^{k-j} e^{cx}$$
$$= x^{-N}e^{cx}(c^k x^k + P_{k-1}(x)),$$

where P_{k-1} is a polynomial of degree $k-1$. Hence, there is an $\tilde{a} \geqslant 0$ such that

$$(x^{k-N}e^{cx})^{(k)} > 0 \qquad \text{for} \quad k = 1, 2, \ldots, n \text{ and every } x > \tilde{a}.$$

Let $\tilde{a} < a < b < \infty$, $1 \leqslant p \leqslant \infty$. By Theorem 6, $\{x^{N-n}, x^{N-n+1}, \ldots, x^{N-1}\}$ is the unique best partial basis of $\{x^{j-1}\}_{j=1}^{N}$ of length n to approximate e^{cx} in the $L^p(a, b)$ norm.

4. Generalizations to ECT-Systems

Let I be a real interval and $f_1, f_2, \ldots, f_n$ functions in $C^{n-1}(I)$, the set of real functions having a continuous $(n-1)$th derivative at each point of I. The sequence $\{f_1, f_2, \ldots, f_n\}$ is called an *extended complete Tchebycheff system* or an *ECT-system on I* iff, for $k = 1, 2, \ldots, n$, the following property holds: If $x_1 \leqslant x_2 \leqslant \cdots \leqslant x_k$ and if $x_j \in I$ for $j = 1, 2, \ldots, k$, then the determinant of the $k \times k$ matrix whose jth row $(j = 1, 2, \ldots, k)$ is

$$f_j^{(1-r_1)}(x_1) \quad f_j^{(2-r_2)}(x_2) \quad \cdots \quad f_j^{(k-r_k)}(x_k)$$

is positive. For $j = 1, 2, \ldots, k$, we denote by r_j the smallest integer r for which $x_r = x_j$.

Let $-\infty < a < b < \infty$ and let $u_0, u_1, \ldots, u_m$ $(m \geqslant 0)$ be real functions in $C^m[a, b]$. Then [5, p. 376], $\{u_k\}_0^m$ is an ECT-system on $[a, b]$ iff, for each $k = 0, 1, \ldots, m$ and each $x \in [a, b]$,

$$W(u_0, \ldots, u_k)(x) = \begin{vmatrix} u_0(x) & u_0'(x) & \cdots & u_0^{(k)}(x) \\ u_1(x) & u_1'(x) & \cdots & u_1^{(k)}(x) \\ \vdots & \vdots & & \vdots \\ u_k(x) & u_k'(x) & \cdots & u_k^{(k)}(x) \end{vmatrix} > 0. \tag{5}$$

If $m \geqslant 1$, then $\{u_k\}_0^m$ is an ECT-system on $[a, b]$ satisfying

$$u_k^{(p)}(a) = 0, \qquad p = 0, 1, \dots, k-1; \quad k = 1, 2, \dots, m \tag{6}$$

iff [5, pp. 378–379], for $k = 0, 1, \dots, m$, there is a function $w_k(x) \in C^{m-k}[a, b]$, positive on $[a, b]$, such that, throughout $[a, b]$,

$$\left.\begin{aligned}
u_0(x) &= w_0(x),\\
u_1(x) &= w_0(x) \int_a^x w_1(x_1)\, dx_1,\\
u_2(x) &= w_0(x) \int_a^x w_1(x_1) \int_a^{x_1} w_2(x_2)\, dx_2\, dx_1,\\
&\;\vdots\\
u_m(x) &= w_0(x) \int_a^x w_1(x_1) \int_a^{x_1} w_2(x_2) \cdots \int_a^{x_{m-1}} w_m(x_m)\, dx_m \cdots dx_1
\end{aligned}\right\} \tag{7}$$

in which case, by (5),

$$u_0(a)\, u_1'(a) \cdots u_m^{(m)}(a) = W(u_0, \dots, u_m)(a) > 0$$

so that

$$u_k^{(k)}(a) \neq 0, \qquad k = 0, 1, \dots, m$$

and, in fact, $u_0(a) = w_0(a) > 0$ and, for $k = 1, 2, \dots, m$,

$$u_k^{(k)}(a) = W(u_0, \dots, u_k)(a) / W(u_0, \dots, u_{k-1})(a) > 0.$$

THEOREM 9. *Let $-\infty < a < b < \infty$, let $w_0, w_1, \dots, w_m$ $(m \geqslant 1)$ be positive functions, $w_k \in C^{m-k+1}[a, b]$, $k = 0, 1, \dots, m$. For $k = 0, 1, \dots, m-1$, let*

$$D_k = \frac{d}{dx} \frac{1}{w_k(x)}. \tag{8}$$

Let $0 \leqslant n < m$, n an integer; $1 \leqslant p \leqslant \infty$. Suppose f is positive and continuous on $(a, b]$ and

$$(-1)^{k+1} D_k \cdots D_0 f > 0 \qquad \text{on } (a, b) \text{ for } k = 0, \dots, n. \tag{9}$$

Consider, on $[a, b]$, the functions (7). Then every subsequence of $\{f, u_0, \dots, u_m\}$ of length $\leqslant n+2$ is a T-system on $(a, b]$. Hence, by Theorem 1(b), if $a < a^* < b$, then $\{u_j\}_0^n$ is the unique best partial basis of $\{u_j\}_0^m$ of length $n+1$ to approximate f in the $L^p(a^*, b)$ norm.

Proof. Every subsequence of $\{u_0,..., u_m\}$ is a T-system on $(a, b]$ [2, Lemma 8, p. 95]. Hence, by Theorem 2(B), it is enough to prove that $\{f, u_0,..., u_k\}$ is a T-system on $(a, b]$ for $k = 0, 1,..., n$. Let $0 \leqslant k \leqslant n$, $t \in [0, 1]$. Assume there are reals $\alpha, \alpha_0,..., \alpha_k$, not all 0, such that

$$q(x) \equiv \alpha[tf(x) + (1-t)(-1)^{k+1} u_{k+1}(x)] + \sum_{r=0}^{k} \alpha_r u_r(x)$$

vanishes at some $k+2$ (distinct) points of $(a, b]$. By a repeated application of Rolle's theorem, we see that, somewhere in (a, b),

$$D_k \cdots D_1 D_0 q(x) = \alpha[tD_k \cdots D_0 f(x) + (1-t)(-1)^{k+1} w_{k+1}(x)] = 0$$

and, hence, $\alpha = 0$. So $\sum_{r=0}^{k} \alpha_r u_r$ vanishes at $k+2$ (distinct) points of $(a, b]$, contradicting the first sentence of this proof. Thus, if $a < x_1 < x_2 \cdots < x_{k+2} \leqslant b$, then

$$\Delta(t) \equiv t \begin{vmatrix} f(x_1) & \cdots & f(x_{k+2}) \\ u_0(x_1) & \cdots & u_0(x_{k+2}) \\ \vdots & & \vdots \\ u_k(x_1) & \cdots & u_k(x_{k+2}) \end{vmatrix} + (1-t)(-1)^{k+1} \begin{vmatrix} u_{k+1}(x_1) & \cdots & u_{k+1}(x_{k+2}) \\ u_0(x_1) & \cdots & u_0(x_{k+2}) \\ \vdots & & \vdots \\ u_k(x_1) & \cdots & u_k(x_{k+2}) \end{vmatrix}$$

never vanishes on $[0, 1]$. As $\Delta(0)$ is positive, so is $\Delta(1)$, which completes the proof.

Instead of dealing, as in Theorem 9, with the functions (7), we can start with a given ECT-system.

THEOREM 10. *Let $-\infty < a < b < \infty$; $0 \leqslant n < m$, n, m integers, $1 \leqslant p \leqslant \infty$. Let $U_0, U_1,..., U_m$ belong to $C^m[a, b]$ and let $\{U_j\}_0^m$ be an ECT-system on $[a, b]$. Suppose every "gapless" subsequence of $\{U_j\}_0^m$ of length $\leqslant n+2$ is a T-system on $[a, b]$. With the notation* (5), *set*

$$\left.\begin{aligned} &w_0 = U_0, \qquad w_1 = W(U_0, U_1)\, U_0^{-2}, \\ &w_k = W(U_0,..., U_{k-2})\, W(U_0,..., U_k)\, W^{-2}(U_0,..., U_{k-1}) \quad \text{if } 2 \leqslant k \leqslant m, \end{aligned}\right\} \tag{10}$$

so that, for $k = 0, 1,..., m$, w_k is positive on $[a, b]$. For $k = 0, 1,..., m-1$, let (8). *Suppose f is positive and continuous on $[a, b]$ and* (9). *Then every subsequence of $\{f, U_0,..., U_m\}$ of length $\leqslant n+2$ is a T-system on $[a, b]$. Hence, by Theorem* 1(b), *$\{U_j\}_0^n$ is the unique best partial basis of $\{U_j\}_0^m$ of length $n+1$ to approximate f in the $L^p(a, b)$ norm.*

Proof. By subtracting from U_k a suitable linear combination of $U_0,..., U_{k-1}$ $(k = 1, 2,..., m)$ and setting $u_0 = U_0$, we obtain [5, p. 379] a sequence $\{u_k\}_{k=0}^m$ which is an ECT-system on $[a, b]$ satisfying (6). It turns out [5, p. 380] that, with (10), we have (7) throughout $[a, b]$. Each U_k $(k=1,\dots, m)$ is obtained from u_k by subtracting from it a suitable linear combination of $u_0,\dots, u_{k-1}$. Hence, for $k=0, 1,\dots, n$, we have, throughout (a, b),

$$D_k \cdots D_1 D_0 U_{k+1} = D_k \cdots D_1 D_0 u_{k+1} = w_{k+1}.$$

We can now imitate the proof of Theorem 9.

Remark 11. Let $-\infty < a < b < \infty$ and let $w_0, w_1,\dots, w_k$ $(k \geqslant 0)$ be real functions, differentiable and positive on (a, b). For $j=0, 1, 2,\dots, k$, let

$$D_j = \frac{d}{dx}\frac{1}{w_j}, \qquad r_j = \sum_{h=0}^{j} \frac{w_h'}{w_h} = \frac{d}{dx}\log(w_0 \cdots w_j).$$

Then (compare [2, p. 92]), for every real function y defined on (a, b), one has

$$(D-r_k)\cdots(D-r_0)\, y = w_0 \cdots w_k D_k \cdots D_0\, y,$$

in the following sense: the left-hand side exists throughout (a, b) iff the right-hand side does, in which case both are equal throughout (a, b).

In particular, it follows that (9) in Theorems 9 and 10 can be replaced by:

$$(-1)^{k+1}(D-r_k)\cdots(D-r_0) f > 0 \qquad \text{on } (a, b) \text{ for } k=0,\dots, n.$$

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